SOLUTIONS
MID-SEMESTER EXAM B-MATH-III 2015-2016
(DIFFERENTIAL TOPOLOGY)

Solution 1(a): True. If $f : S^1 \times S^1 \to S^2$ is an immersion then because of dimension count, it is also submersion. By Submersion Theorem, it is a local diffeomorphism. Hence the image $f(S^1 \times S^1)$ is an open subset of S^2. As $S^1 \times S^1$ is compact, the image $f(S^1 \times S^1)$ is also closed subset of S^2. This implies that $f(S^1 \times S^1) = S^2$. Now using the fact that $S^1 \times S^1$ is compact, we conclude that f is a covering map. But, as S^2 is simply connected, we get a contradiction.

Solution 1(b): True. Consider the standard inclusion $i : S^1 \times S^1 \to \mathbb{R}^3$. As $\mathbb{R}^3$ is a submanifold of $\mathbb{R}^5$, and $\mathbb{R}^5$ is an open subset of S^5, we compose all these inclusion to get a 1-1 inclusion of $S^1 \times S^1$ into S^5.

Solution 1(c): True. As $f : X \to Y$ is a diffeomorphism, for an $p \in X$, the derivative map $D_p(f) : T_pX \to T_{f(p)}Y$ is an isomorphism. Hence, it is now easy to see that f is transversal to any submanifold Z of Y. Thus is true irrespective to the dimension of Z.

Solution 1(d): True. If $f : S^2 \to S^1$ do not have any critical point then the map f is a submersion. That means, for any point $p \in S^2$, the derivative map $D_pf : T_pS^2 \to T_{f(p)}S^1$ is surjective. We denote the kernel of this homomorphsim by K_p. Next we induce the standard Reimannian metric on S^2. As the tangent bundle of S^1 is trivial, there exits a nowhere vanishing vector field ξ on S^1. We now define a vector field μ on S^2 such that μ_p is orthogonal to K_p and $D_pf(\mu_p) = \xi_{f(p)}$. Note that μ is a nowhere vanishing vector field on S^2. This is a contradiction.

Solution 1(e): True. Let X, Z and Y be of dimension m,n and k, respectively. Let $y \in X \bigcap Z$. We can choose a coordinate neighborhood U around y in Y such that $X \bigcap U$ is the zero set of a regular function $f : U \to \mathbb{R}^{k-m}$ and $Z \bigcap U$ is the zero set of a regular function $g : U \to \mathbb{R}^{k-n}$. Consider the function $F : U \to \mathbb{R}^{2k-(m+n)}$ given by $F(x, y) = (f(x), g(x))$. Using the fact that X and Z are transversal, we can check that F is a smooth regular function. Therefore, $X \bigcap Z$ is $k - (m + n)$ dimension submanifold of Y. Since, $\dim(T_yX \bigcap T_yZ) = k - \dim T_yX - \dim T_yZ$, and $T_y(X \bigcap Z) \subseteq T_yX \bigcap T_yZ$, we conclude that

$$T_y(X \bigcap Z) = T_yX \bigcap T_yZ.$$

Solution 2(a): Let $S_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A = A^T\}$ be the set of $(n \times n)$ symmetric matrices. Consider the map $\phi : M_n(\mathbb{R}) \to S_n(\mathbb{R})$ by $\phi(A) = AA^T$. Now the derivative $D_A\phi : M_n(\mathbb{R}) \to S_n(\mathbb{R})$ given by

$$D_A\phi(B) = \lim_{s\to 0} \frac{\phi(A + sB) - \phi(A)}{s} = BA^T + AB^T$$

is surjective. Therefore, the map ϕ is submersion. This implies that $O(n) = \phi^{-1}(I)$ is a submanifold of $M_n(\mathbb{R})$ of dimension $n^2 - n(n+1)/2 = n(n-1)/2$.

Solution 3(a): Construct a smooth function $f : \mathbb{R} \to \mathbb{R}$ with support in $[-1/2, 1/2]$ satisfying $f(0) = 1, f'(0) = 0$. Let $g : \mathbb{Z} \to \mathbb{Q}$ be a bijection. Then the set of critical points of the function

$$\sum_{n \in \mathbb{N}} g(n) f(x - n)$$

contains $\mathbb{N}$, and $f(\mathbb{N}) = \mathbb{Q}$.

Solution 3(b): Consider the following two coordinate charts on S^1: $f_1 : (0, 2\pi) \to \mathbb{R}^2$ given by

$$f_1(t) = (\cos t, \sin t),$$

and $f_2 : (0, 2\pi) \to \mathbb{R}^2$ given by

$$f_2(t) = (-\cos t, -\sin t).$$

Hence any critical points (x, y) of the function f is of the form $f_i(t_i)$, where t_i is a critical points of the function $f \circ f_i$ for $i = 1, 2$. A point t_1 is a critical point of $f \circ f_1$ if and only if the derivative $\frac{\partial (f \circ f_1)}{\partial t}(t_1) = 0$. This means

$$(\cos t_1)^2 - (\sin t_1)^2 = 0$$

This implies that $\pi/4$, $3\pi/4$, $5\pi/4$ and $7\pi/4$ are four critical points of $f \circ f_1$. Also observe that the second derivative $\frac{\partial^2 (f \circ f_1)}{\partial t}$ is non-zero at all the above critical points. Hence all these critical points are non-degenerate. Similarly, we can show that $\pi/4$, $3\pi/4$, $5\pi/4$ and $7\pi/4$ are four critical points of $f \circ f_2$ and all of these are non-degenerate. Hence the four critical points of the function f are:

$$(\cos \pi/4, \sin \pi/4), (\cos 3\pi/4, \sin 3\pi/4), (\cos 5\pi/4, \sin 5\pi/4) \text{ and } (\cos 7\pi/4, \sin 7\pi/4).$$

All of these critical points are non-degenerate.